

The Design for Proofs using Division into Cases:
(For three (3) cases here)

To Prove: Statement p .

Proof: . . . (statements defining variables and establishing
initial results) . . .

$\therefore q$ OR r OR s by

[Required List of all possible cases]

CASE 1: (statement q)

[Required Case Heading]

Suppose q .

. . .
. . .

$\therefore p$ in the case that q . [or " $\therefore p$ in Case 1."]

[Case 1 Conditional Conclusion]

CASE 2: (statement r)

[Required Case Heading]

Suppose r .

. . .
. . .

$\therefore p$ in the case that r . [or " $\therefore p$ in Case 2."]

[Case 2 Conditional Conclusion]

CASE 3: (statement s)

[Required Case Heading]

Suppose s .

. . .
. . .

$\therefore p$ in the case that s . [or " $\therefore p$ in Case 3."]

[Case 3 Conditional Conclusion]

[Since p has been proved true in all possible cases, we can conclude p is true.]

$\therefore p$ in general.

[The Conclusion Without Conditions]

QED

On the following pages is an example of a proof which uses Division into Cases.

The commentary provided is important to read.

An Example of a Proof using Division into Cases

To Prove: For every integer n , $n^2 + 5n + 7$ is odd. [The proof is presented with added commentary.]

Proof: Let n be any integer. [NTS: $n^2 + 5n + 7$ is odd.]

[The “Parity Corollary” states: For every integer k , k is even or k is odd.]

By the Parity Corollary, n is even or n is odd. [The required statement of all possible cases to consider.]

Case 1: (n is even.)

[Introduce the case argument with the correct heading.]

Suppose that n is even.

By definition of “even”, there exists an integer k such that $n = 2k$.

$$\begin{aligned}\therefore n^2 + 5n + 7 &= (2k)^2 + 5(2k) + 7 && \text{by substitution,} \\ &= 4k^2 + 10k + (6+1) \\ &= (4k^2 + 10k + 6) + 1 && \text{by Rules of Algebra.}\end{aligned}$$

$$\therefore n^2 + 5n + 7 = 2(2k^2 + 5k + 3) + 1 \quad \text{by Rules of Algebra.}$$

Let $t = 2k^2 + 5k + 3$, which is an integer since sums and products of integers are integers.

$$\therefore n^2 + 5n + 7 = 2t + 1, \text{ by substitution, and } t \text{ is an integer.}$$

$$\therefore n^2 + 5n + 7 \text{ is odd,}$$

by definition of “odd”, in the case that n is even. [The Case 1 Conditional Conclusion]

[An alternate way to state the Case 1 Conditional Conclusion is as follows:

“ $\therefore n^2 + 5n + 7$ is odd, by definition of “odd”, in Case 1. ”]

[End of Case 1]

Case 2: (n is odd.)

Suppose that n is odd.

By definition of “odd”, $n = 2k + 1$ for some integer k .

$$\begin{aligned}\therefore n^2 + 5n + 7 &= (2k + 1)^2 + 5(2k + 1) + 7 && \text{by substitution,} \\ &= (4k^2 + 4k + 1) + (10k + 5) + 7 \\ &= (4k^2 + 14k + 12) + 1 \\ &= 2(2k^2 + 7k + 6) + 1 && \text{by Rules of Algebra}\end{aligned}$$

[Recall the last statement of the proof of the previous page:

$$\therefore n^2 + 5n + 7 = \dots = 2(2k^2 + 7k + 6) + 1 \quad \text{by Rules of Algebra]}$$

[Continuing the proof:]

$$\therefore n^2 + 5n + 7 = 2t + 1, \text{ where } t = (2k^2 + 7k + 6), \text{ which is an integer.}$$

$\therefore n^2 + 5n + 7$ is odd,
by definition of “odd”, in the case that n is odd. [The Case 2 Conditional Conclusion]

[An alternate way to state the Case 2 Conditional Conclusion is as follows:

“ $\therefore n^2 + 5n + 7$ is odd, by definition of “odd”, in Case 2. ”]

[End of Case 2]

[Because the statement has been proved true in all possible cases, we can now conclude:]

$\therefore n^2 + 5n + 7$ is odd in general. [The Conclusion Without Conditions (Required)]

\therefore For every integer n , $n^2 + 5n + 7$ is odd, by Direct Proof. [This is a Direct Proof of a Universal Stmt.]

QED

Note: None of the comments appearing in the proof above are required.

It is RECOMMENDED that the comments [End of Case 1] and [End of Case 2] be used.

The headings “Case 1: (...)”, “Case 2: (...)”, etc, are REQUIRED.

The case heading should be followed immediately by a statement which assumes the condition restricting the scope of the argument. The author uses the convention that the statement of the condition in the case heading makes this assumption automatically, a convention which is acceptable to use in this class.

The algebraic derivation of the formula for t (by repeated applications of the Rules of Algebra) must be presented in several steps, with sufficient simplicity that the reader does not need to make significant effort to verify the derivation.

For example, the following is not acceptable:

$$\begin{aligned} \therefore n^2 + 5n + 7 &= (2k + 1)^2 + 5(2k + 1) + 7 && \text{by substitution,} \\ &= 2(2k^2 + 7k + 6) + 1 && \text{by Rules of Algebra.} \end{aligned}$$

On the following page, this proof is presented again, omitting most of the comments made above:

To Prove: For every integer n , $n^2 + 5n + 7$ is odd.

Proof: Let n be any integer. [NTS: $n^2 + 5n + 7$ is odd .]

By the Parity Corollary, n is even or n is odd.

Case 1: (n is even.)

Suppose that n is even.

By definition of “even”, there exists an integer k such that $n = 2k$.

$$\begin{aligned}\therefore n^2 + 5n + 7 &= (2k)^2 + 5(2k) + 7 && \text{by substitution,} \\ &= 4k^2 + 10k + (6 + 1) \\ &= (4k^2 + 10k + 6) + 1 && \text{by Rules of Algebra.}\end{aligned}$$

$$\therefore n^2 + 5n + 7 = 2(2k^2 + 5k + 3) + 1 \quad \text{by Rules of Algebra.}$$

Let $t = 2k^2 + 5k + 3$, which is an integer since sums and products of integers are integers.

$$\therefore n^2 + 5n + 7 = 2t + 1, \text{ by substitution, and } t \text{ is an integer.}$$

$\therefore n^2 + 5n + 7$ is odd, by definition of “odd” in the case that n is even. [End of Case 1]

Case 2: (n is odd.)

Suppose that n is odd.

By definition of “odd”, $n = 2k + 1$ for some integer k .

$$\begin{aligned}\therefore n^2 + 5n + 7 &= (2k + 1)^2 + 5(2k + 1) + 7 && \text{by substitution,} \\ &= (4k^2 + 4k + 1) + (10k + 5) + 7 \\ &= (4k^2 + 14k + 12) + 1 \\ &= 2(2k^2 + 7k + 6) + 1 && \text{by Rules of Algebra}\end{aligned}$$

$$\therefore n^2 + 5n + 7 = 2t + 1, \text{ where } t = (2k^2 + 7k + 6), \text{ and } t \text{ is an integer.}$$

$\therefore n^2 + 5n + 7$ is odd, by definition of “odd” in the case that n is odd. [End of Case 2]

$\therefore n^2 + 5n + 7$ is odd in general.

\therefore For every integer n , $n^2 + 5n + 7$ is odd, by Direct Proof.

QED